

# TOPOLOGICAL CORRESPONDENCE OF MULTIPLE ERGODIC AVERAGES OF NILPOTENT GROUP ACTIONS

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**ABSTRACT.** Let  $(X, \Gamma)$  be a topological system, where  $\Gamma$  is a nilpotent group generated by  $T_1, \dots, T_d$  such that for each  $T \in \Gamma$ ,  $T \neq e_\Gamma$ ,  $(X, T)$  is weakly mixing and minimal. For  $d, k \in \mathbb{N}$ , let  $p_{i,j}(n)$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq d$  be polynomials with rational coefficients taking integer values on the integers and  $p_{i,j}(0) = 0$ . We show that if the expressions  $g_i(n) = T_1^{p_{i,1}(n)} \dots T_d^{p_{i,d}(n)}$  depends nontrivially on  $n$  for  $i = 1, 2, \dots, k$ , and for all  $i \neq j \in \{1, 2, \dots, k\}$  the expressions  $g_i(n)g_j(n)^{-1}$  depend nontrivially on  $n$ , then there is a residual set  $X_0$  of  $X$  such that for all  $x \in X_0$

$$\{(g_1(n)x, g_2(n)x, \dots, g_k(n)x) \in X^k : n \in \mathbb{Z}\}$$

is dense in  $X^k$ .

## 1. INTRODUCTION

Measurable dynamics and topological dynamics are two sister branches of the theory of dynamical systems, who use similar words to describe different but parallel notions in their respective theories. The surprising fact is that many of the corresponding results are rather similar though the proofs may be quite different. For the interplay between measurable and topological dynamics, we refer to the survey by Glasner and Weiss [13]. In this paper, we study the topological analogue of multiple ergodic averages of weakly mixing systems under nilpotent group actions.

### 1.1. Main results.

Motivated by the work of Furstenberg on the multiple recurrence theorem [7], in his pioneer work Glasner presented in [12] the counterpart of [7] in topological dynamics. As it is said in [12]: “The basic problem in both the measure theoretical and the topological theory is roughly the following: given a system  $(X, T)$  (ergodic or minimal) and a positive integer  $n$ , describe the most general relation that holds for  $(n+1)$ -tuples  $(x, Tx, T^2x, \dots, T^n x)$  in the product space  $X \times X \times \dots \times X$  ( $n+1$  times).” One of the main results in [12] is that: for a topologically weakly mixing and minimal system  $(X, T)$ , there is a dense  $G_\delta$  subset  $X_0$  such that for each  $x \in X_0$ ,  $(T^n x, \dots, T^{dn} x)$  is dense in  $X^d$ . Note that a different proof of Glasner’s theorem on weakly mixing systems was presented in [17, 20].

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In this paper we extend this result to a much broader setting. Let  $\mathcal{P}$  be the collection of all polynomials with rational coefficients taking integer values on the integers,  $\mathcal{P}_0$  be the collection of elements  $p$  of  $\mathcal{P}$  with  $p(0) = 0$ , and  $\mathcal{P}_0^*$  be the collection of non-constant elements of  $\mathcal{P}_0$ . The main results of this paper are the following:

**Theorem 1.1.** *Let  $(X, \Gamma)$  be a topological system, where  $\Gamma$  is a nilpotent group such that for each  $T \in \Gamma$ ,  $T \neq e_\Gamma$ , is weakly mixing and minimal. For  $d, k \in \mathbb{N}$  let  $T_1, \dots, T_d \in \Gamma$ ,  $\{p_{i,j}(n)\}_{1 \leq i \leq k, 1 \leq j \leq d} \in \mathcal{P}_0$  such that the expression*

$$g_i(n) = T_1^{p_{i,1}(n)} \dots T_d^{p_{i,d}(n)}$$

*depends nontrivially on  $n$  for  $i = 1, 2, \dots, k$ , and for all  $i \neq j \in \{1, 2, \dots, k\}$  the expressions  $g_i(n)g_j(n)^{-1}$  depend nontrivially on  $n$ . Then there is a dense  $G_\delta$  subset  $X_0$  of  $X$  such that for all  $x \in X_0$*

$$\{(g_1(n)x, \dots, g_k(n)x) : n \in \mathbb{Z}\}$$

*is dense in  $X^k$ .*

We remark that the non-degeneracy conditions stated in the above theorem is also necessary. Note that we say that  $g(n)$  depends nontrivially on  $n$ , if  $g(n)$  is a nonconstant mapping from  $\mathbb{Z}$  into  $\Gamma$ , and  $g_1(n), g_2(n)$  are distinct if  $g_1(n)g_2^{-1}(n)$  depends nontrivially on  $n$ . When  $\Gamma$  is abelian, one has that

$$g_i(n)g_j(n)^{-1} = T_1^{p_{i,1}(n)-p_{j,1}(n)} \dots T_d^{p_{i,d}(n)-p_{j,d}(n)}.$$

When  $\Gamma$  is nilpotent, the expressions of  $g_i(n)$  and  $g_i(n)g_j(n)^{-1}$  depend on the Malcev basis of  $\Gamma$  (see Section 3).

Taking  $\Gamma = \mathbb{Z}$  and  $d = 1$  in Theorem 1.1, we have the result for one transformation.

**Theorem 1.2.** *Let  $(X, T)$  be a weakly mixing minimal system and  $p_1, \dots, p_d \in \mathcal{P}_0^*$  be distinct polynomials. Then there is a dense  $G_\delta$  subset  $X_0$  of  $X$  such that for any  $x \in X_0$*

$$\{(T^{p_1(n)}(x), \dots, T^{p_d(n)}(x)) : n \in \mathbb{Z}\}$$

*is dense in  $X^d$ .*

## 1.2. Multiple ergodic averages for weakly mixing systems.

Now we state some corresponding results in ergodic theory. For a weakly mixing system, Bergelson and Leibman [2, Theorem D] showed the following result: Let  $(X, \mathcal{X}, \mu, \Gamma)$  be a measure preserving system, where  $\Gamma$  is an abelian group such that for each  $T \in \Gamma$ ,  $T \neq e_\Gamma$ , is weakly mixing. For  $d, k \in \mathbb{N}$ , let  $T_1, \dots, T_d \in \Gamma$ , and  $p_{i,j} \in \mathcal{P}_0$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq d$  such that the expressions  $g_i(n)$  satisfies the non-degeneracy conditions stated in Theorem 1.1. Then for any  $f_1, \dots, f_k \in L^\infty(X, \mu)$ ,

$$\lim_{N \rightarrow \infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} \prod_{i=1}^k f_i(T_1^{p_{i,1}(n)} \dots T_d^{p_{i,d}(n)} x) - \prod_{i=1}^k \int_X f_i(x) d\mu \right\|_{L^2} = 0.$$

Related results were proved for nilpotent group actions by Leibman [19, Theorem 11.15]. Note that topological and measurable multiple recurrent theorems under nilpotent group actions were also studied in [2, 18, 24].

It is natural to conjecture that the result above is still valid for the pointwise convergence, i.e. for weakly mixing nilpotent group actions, we conjecture that for a subset  $X_0$  with full measure, and each  $x \in X_0$  the averages

$$\frac{1}{N} \sum_{n=1}^N \prod_{j=1}^k f_j(T_1^{p_{1,j}(n)} T_2^{p_{2,j}(n)} \dots T_d^{p_{d,j}(n)} x)$$

converge to the product of the integrals if  $g_i(n) = T_1^{p_{i,1}(n)} \dots T_d^{p_{i,d}(n)}$ ,  $i = 1, 2, \dots, k$  satisfy the obvious non-degeneracy condition. In this paper in some sense we add an evidence to support this conjecture, i.e. we give a topological correspondence of multiple ergodic averages of nilpotent weakly mixing group actions.

Finally we say a few more words on multiple ergodic averages. Followed from Furstenberg's beautiful work [7] on the dynamical proof of Szemerédi's theorem in 1977, problems concerning the convergence of multiple ergodic averages (or called "non-conventional averages" [9, 10]) in  $L^2$  or pointwisely attract a lot of attention. Nowadays we have rich results for the  $L^2$ -norm convergence [14, 21, 22, 23]. On the other hand, there are a few results related to the pointwise convergence of multiple ergodic averages. Bourgain showed that the limit of  $\frac{1}{N} \sum_{n=0}^{N-1} f(T^{p(n)} x)$  exists a.e. for all integer valued polynomials  $p(n)$  and  $f \in L^p(X, \mathcal{X}, \mu)$  with  $p > 1$  [3], and the averages  $\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{a_1 n} x) f_2(T^{a_2 n} x)$  converge a.e. for  $a_1, a_2 \in \mathbb{Z}$  and all  $f_1, f_2$  in  $L^\infty(X, \mathcal{X}, \mu)$ . Huang, Shao and Ye [16] showed that for all distal systems,  $\frac{1}{N} \sum_{n=0}^{N-1} f_1(T^n x) \dots f_d(T^{dn} x)$  exists a.e. for all  $f_1, \dots, f_d \in L^\infty(X, \mathcal{X}, \mu)$ , where  $d \in \mathbb{N}$ . Very recently, Donoso and Sun in [5] generalized the above result to commuting distal transformations.

In [12] it was also showed that, up to a canonically defined proximal extension, a characteristic family for  $T \times T^2 \times \dots \times T^n$ , is the family of canonical PI flows of class  $n-1$ . In particular, when  $(X, T)$  is minimal and distal, most  $T \times T^2 \times \dots \times T^n$  orbit closures of points  $(x, x, \dots, x)$  in the diagonal of  $X^n$  are lifts of the corresponding orbit closures in the largest class- $(n-1)$  factor. In view of this fact and the recent progress related to the convergence of multiple ergodic averages, it is an interesting question how to formulate and prove the counterpart in topological dynamics for nilpotent group actions. In this paper we have investigated the weak mixing case, and we plan to treat the non-weakly mixing system in the future research.

### 1.3. Strategy of the proofs and further results.

To prove Theorem 1.1 we use PET-induction, which was introduced by Bergelson in [1]. The PET-induction we use in the current paper is due to Leibman [18]. The basic idea of this induction is that: we associate any finite collection of polynomials a "complexity", and reduce the complexity at some step to the trivial one. Note that in some step, the cardinal number of the collection may increase while the complexity decreases.

It is easy to show that to prove Theorem 1.1, it is equivalent to prove that for any given non-empty open subsets  $U, V_1, \dots, V_k$  of  $X$ ,

$$(1.1) \quad \{n \in \mathbb{Z} : U \cap (g_1(n)^{-1} V_1 \cap \dots \cap g_k(n)^{-1} V_k) \neq \emptyset\}$$

is infinite (Lemma 2.4). Basically, this can be done by proving a proposition related to the weakly mixing property (Lemma 2.6) and the fact that for all non-empty open sets  $U_1, \dots, U_k$  and  $V_1, \dots, V_k$  of  $X$

$$(1.2) \quad \{n \in \mathbb{Z} : U_1 \times \dots \times U_k \cap g_1(n)^{-1} \times \dots \times g_k(n)^{-1}(V_1 \times \dots \times V_k) \neq \emptyset\}$$

is infinite. Practically, when doing this, we find that if in the collection of polynomials there are linear elements and other non-linear elements, the argument will be very much involved. To overcome this difficulty, we actually show that for non-empty open subsets  $U, V$  and a  $\Gamma$ -polynomial  $g(n)$ ,  $\{n \in \mathbb{Z} : U \cap g(n)^{-1}(V) \neq \emptyset\}$  is thickly-syndetic. Since the family of thickly-syndetic subsets is a filter, this implies (1.2). To prove this, we need to show that (1.1) is syndetic. This means that the proof of Theorem 1.1 is achieved by showing the following stronger result:

**Theorem 1.3.** *Let  $(X, \Gamma)$  be a topological system, where  $\Gamma$  is a nilpotent group such that for each  $T \in \Gamma$ ,  $T \neq e_\Gamma$ , is weakly mixing and minimal. For  $d, k \in \mathbb{N}$  let  $T_1, \dots, T_d \in \Gamma$ ,  $\{p_{i,j}(n)\}_{1 \leq i \leq k, 1 \leq j \leq d} \in \mathcal{P}_0$  such that the expression*

$$g_i(n) = T_1^{p_{i,1}(n)} \dots T_d^{p_{i,d}(n)}$$

*depends nontrivially on  $n$  for  $i = 1, 2, \dots, k$ , and for all  $i \neq j \in \{1, 2, \dots, k\}$  the expressions  $g_i(n)g_j(n)^{-1}$  depend nontrivially on  $n$ . Then for all non-empty open sets  $U_1, \dots, U_k$  and  $V_1, \dots, V_k$  of  $X$*

$$\{n \in \mathbb{Z} : U_1 \times \dots \times U_k \cap g_1(n)^{-1} \times \dots \times g_k(n)^{-1}(V_1 \times \dots \times V_k) \neq \emptyset\}$$

*is a thickly-syndetic set, and*

$$\{n \in \mathbb{Z} : U \cap (g_1(n)^{-1}V_1 \cap \dots \cap g_k(n)^{-1}V_k) \neq \emptyset\}$$

*is a syndetic set.*

We note that when doing the induction procedure, we need to check the non-degeneracy conditions of the reduced collection. We find that the known results are not enough to guarantee them, and we should prove additional lemmas whose proofs are presented in Subsection 5.2.2.

After we introduce PET-induction in Section 3, we will explain the main ideas of the proof via proving Theorem 1.2. As an application of Theorem 1.1 we have

**Theorem 1.4.** *Let  $(X, \Gamma)$  be a topological system, where  $\Gamma$  is a nilpotent group such that for each  $T \in \Gamma$ ,  $T \neq e_\Gamma$ , is weakly mixing and minimal. For  $k \in \mathbb{N}$  let  $T_1, \dots, T_k \in \Gamma$ ,  $\{p_i(n)\}_{1 \leq i \leq k} \in \mathcal{P}_0$  such that the expression  $g(n) = T_1^{p_1(n)} \dots T_k^{p_k(n)}$  depends nontrivially on  $n$ . Then there is a dense  $G_\delta$  subset  $X_0$  of  $X$  such that for each  $x \in X_0$  and each non-empty open subset  $U$  of  $X$*

$$N_g(x, U) := \{n \in \mathbb{Z} : g(n)x \in U\}$$

*is piecewise syndetic.*

**Remark 1.5.** It is easy to see that to show the above theorems, we may assume that the coefficients of the polynomials involved are integers.

**1.4. Organization of the paper.** We organize the paper as follows. In Section 2 we introduce some basic notions and facts we need in the paper. In Section 3, we recall the PET-induction for nilpotent group actions. In Section 4, we show some examples and outline the proof of Theorem 1.2, which provides the main ideas how to prove Theorem 1.3. In the final section, we give the complete proof of Theorem 1.3 and Theorem 1.4.

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## 2. PRELIMINARY

### 2.1. Topological transformation groups.

A *topological dynamical system* (t.d.s. for short) is a triple  $\mathcal{X} = (X, \Gamma, \Pi)$ , where  $X$  is a compact metric space,  $\Gamma$  is a Hausdorff topological group with the unit  $e_\Gamma$  and  $\Pi : \Gamma \times X \rightarrow X$  is a continuous map such that  $\Pi(e_\Gamma, x) = x$  and  $\Pi(s, \Pi(t, x)) = \Pi(st, x)$ . We shall fix  $\Gamma$  and suppress the action symbol. In many references,  $(X, \Gamma)$  is also called a *topological transformation group* or a *flow*.

Let  $(X, \Gamma)$  be a t.d.s. and  $x \in X$ , then  $\mathcal{O}(x, \Gamma)$  denotes the *orbit* of  $x$ , which is also denoted by  $\Gamma x$ . A subset  $A \subseteq X$  is called *invariant* if  $ta \subseteq A$  for all  $a \in A$  and  $t \in \Gamma$ . When  $Y \subseteq X$  is a closed and  $\Gamma$ -invariant subset of the system  $(X, \Gamma)$  we say that the system  $(Y, \Gamma)$  is a *subsystem* of  $(X, \Gamma)$ . If  $(X, \Gamma)$  and  $(Y, \Gamma)$  are two dynamical systems their *product system* is the system  $(X \times Y, \Gamma)$ , where  $t(x, y) = (tx, ty)$ .

A system  $(X, \Gamma)$  is called *minimal* if  $X$  contains no proper non-empty closed invariant subsets.  $(X, \Gamma)$  is called *transitive* if every non-empty invariant open subset of  $X$  is dense. An example of a transitive system is the *point-transitive* system, which is a system with a dense orbit. It is easy to verify that a system is minimal iff every orbit is dense. A point  $x \in X$  is called a *minimal point* if  $(\overline{\mathcal{O}(x, \Gamma)}, \Gamma)$  is a minimal subsystem. A system  $(X, \Gamma)$  is *weakly mixing* if the product system  $(X \times X, \Gamma)$  is transitive.

### 2.2. Some important subsets of integers.

A subset  $S$  of  $\mathbb{Z}$  is *syndetic* if it has bounded gaps, i.e. there is  $N \in \mathbb{N}$  such that  $\{i, i+1, \dots, i+N\} \cap S \neq \emptyset$  for every  $i \in \mathbb{Z}$ .  $S$  is *thick* if it contains arbitrarily long runs of integers, i.e. there is a subsequence  $\{n_i\}_{i=1}^\infty$  of  $\mathbb{Z}$  with  $|n_{i+1}| > |n_i|$  for any  $i \in \mathbb{N}$  such that  $S \supset \bigcup_{i=1}^\infty \{n_i, n_i+1, \dots, n_i+i\}$ . Some dynamical properties can be interrupted by using the notions of syndetic or thick subsets. For example, a classic result of Gottschalk and Hedlund stated that  $x$  is a minimal point if and only if

$$N(x, U) = \{n \in \mathbb{Z} : T^n x \in U\}$$

is syndetic for any neighborhood  $U$  of  $x$ , and by Furstenberg [6] a topological system  $(X, T)$  is weakly mixing if and only if

$$N(U, V) = \{n \in \mathbb{Z} : U \cap T^{-n}V \neq \emptyset\}$$

is thick for any non-empty open subsets  $U, V$  of  $X$ .

A subset  $S$  is called *thickly-syndetic* if for every  $N \in \mathbb{N}$  the positions where length  $N$  runs begin form a syndetic set. A subset  $S$  of  $\mathbb{Z}$  is *piecewise syndetic* if it is an intersection of a syndetic set with a thick set.

Note that the set of all thickly-syndetic sets is a filter, i.e. the intersection of two thickly-syndetic sets is still a thickly-syndetic set (see [8] for more details).

The following lemma will be used in the sequel.

**Lemma 2.1.** [15, Theorem 4.7.] *For a minimal and weakly mixing system  $(X, T)$ ,*

$$N(U, V) = \{n \in \mathbb{Z} : U \cap T^{-n}V \neq \emptyset\}$$

*is thickly-syndetic for any nonempty open subsets  $U, V$  of  $X$ .*

Since the collection of all thickly syndetic sets is a filter, one consequence of Lemma 2.1 is:

**Corollary 2.2.** *Let  $d \in \mathbb{N}$  and  $(X, T_1), \dots, (X, T_d)$  be weakly mixing and minimal systems. Then  $(X_1 \times \dots \times X_d, T_1 \times \dots \times T_d)$  is weakly mixing.*

### 2.3. Some notions and useful lemmas.

#### 2.3.1. Notations.

Let  $(X, \Gamma)$  be a t.d.s.,  $\Gamma$  be a group, and  $d, k \in \mathbb{N}$ . Let  $T_1, \dots, T_d \in \Gamma$ ,  $p_{i,j}(n) \in \mathcal{P}_0$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq d$ , and let

$$g_i(n) = T_1^{p_{i,1}(n)} \dots T_d^{p_{i,d}(n)}, \quad i = 1, 2, \dots, k.$$

We will fix the above notation in the rest of this section.

#### 2.3.2. $\{g_1, \dots, g_k\}_\Delta$ -transitivity.

**Definition 2.3.** We say  $(X, \Gamma)$  is  $\{g_1, \dots, g_k\}_\Delta$ -transitive if for any given non-empty open subsets  $U, V_1, \dots, V_k$  of  $X$ ,

$$\{n \in \mathbb{Z} : U \cap (g_1(n)^{-1}V_1 \cap \dots \cap g_k(n)^{-1}V_k) \neq \emptyset\}$$

is infinite.

The following lemma is a generalization of an observation in [20].

**Lemma 2.4.** *Let  $(X, \Gamma)$  and  $g_1, \dots, g_k$  be defined as in subsection 2.3.1. Then there is a dense  $G_\delta$  set  $X_0$  of  $X$  such that for all  $x \in X_0$*

$$\{(g_1(n)x, g_2(n)x, \dots, g_k(n)x) \in X^k : n \in \mathbb{Z}\}$$

*is dense in  $X^k$  if and only if it is  $\{g_1, \dots, g_k\}_\Delta$ -transitive.*

*Proof.* One direction is obvious. And now assume that for any given non-empty open sets  $U, V_1, \dots, V_k$  of  $X$ ,  $\{n \in \mathbb{Z} : U \cap (g_1(n)^{-1}V_1 \cap \dots \cap g_k(n)^{-1}V_k) \neq \emptyset\}$  is infinite.

Let  $\mathcal{F}$  be a countable base of  $X$ , and let

$$X_0 = \bigcap_{V_1, \dots, V_k \in \mathcal{F}} \bigcup_{n \in \mathbb{Z}} g_1(n)^{-1}V_1 \cap \dots \cap g_k(n)^{-1}V_k.$$

Then it is easy to see that the dense  $G_\delta$  subset  $X_0$  is what we need.  $\square$

Hence by Lemma 2.4, Theorem 1.1 can be restated as: Assume all the conditions in Theorem 1.1, then  $(X, \Gamma)$  is  $\{g_1, \dots, g_k\}_\Delta$ -transitive.

### 2.3.3. $\{g_1, \dots, g_k\}$ -transitivity.

**Definition 2.5.**  $(X, \Gamma)$  is  $\{g_1, \dots, g_k\}$ -transitive if for all non-empty open sets  $U_1, \dots, U_d$  and  $V_1, \dots, V_d$  of  $X$ ,

$$\{n \in \mathbb{Z} : U_1 \times \dots \times U_k \cap g_1^{-1}(n) \times \dots \times g_k^{-1}(n)(V_1 \times \dots \times V_d) \neq \emptyset\}$$

is infinite.

The following lemma is a generalization of Lemma 3 of [17].

**Lemma 2.6.** Let  $(X, \Gamma)$  and  $g_1, \dots, g_k$  be defined as in subsection 2.3.1 and  $T \in \Gamma$ . If  $(X, \Gamma)$  is  $\{g_1, \dots, g_k\}$ -transitive, then for any non-empty open sets  $V_1, \dots, V_k$  of  $X$  and any subsequence  $\{r(n)\}_{n=0}^\infty$  of natural numbers, there is a sequence of integers  $\{k_n\}_{n=0}^\infty$  such that  $|k_0| > r(0)$ ,  $|k_n| > |k_{n-1}| + r(|k_{n-1}|)$  for all  $n \geq 1$ , and for each  $i \in \{1, 2, \dots, k\}$ , there is a descending sequence  $\{V_i^{(n)}\}_{n=0}^\infty$  of non-empty open subsets of  $V_i$  such that for each  $n \geq 0$  one has that

$$g_i(k_j)T^{-j}V_i^{(n)} \subseteq V_i, \quad \text{for all } 0 \leq j \leq n.$$

*Proof.* Let  $V_1, \dots, V_k$  be non-empty open subsets of  $X$ . Since  $(X, \Gamma)$  is  $\{g_1, \dots, g_k\}$ -transitive, there is some  $k_0$  with  $|k_0| > r(0)$  such that

$$V_1 \times \dots \times V_k \cap g_1^{-1}(k_0) \times \dots \times g_k^{-1}(k_0)(V_1 \times \dots \times V_k) \neq \emptyset.$$

That is,  $g_i^{-1}(k_0)V_i \cap V_i \neq \emptyset$  for all  $i = 1, \dots, k$ . Put  $V_i^{(0)} = g_i^{-1}(k_0)V_i \cap V_i$  for all  $i = 1, \dots, k$  to complete the base step.

Now assume that for  $n \geq 1$  we have found numbers  $k_0, k_1, \dots, k_{n-1}$  and for each  $i = 1, \dots, k$ , we have non-empty open subsets  $V_i \supseteq V_i^{(0)} \supseteq V_i^{(1)} \dots \supseteq V_i^{(n-1)}$  such that  $|k_0| > r(0)$ , and for each  $m = 1, \dots, n-1$  one has  $|k_m| > |k_{m-1}| + r(|k_{m-1}|)$  and

$$(2.1) \quad g_i(k_j)T^{-j}V_i^{(m)} \subseteq V_i, \quad \text{for all } 0 \leq j \leq m.$$

For  $i = 1, \dots, k$ , let  $U_i = T^{-n}(V_i^{(n-1)})$ . Since  $(X, \Gamma)$  is  $\{g_1, \dots, g_k\}$ -transitive, there is some  $k_n \in \mathbb{Z}$  such that  $|k_n| > |k_{n-1}| + r(|k_{n-1}|)$  and

$$U_1 \times \dots \times U_k \cap g_1^{-1}(k_n) \times \dots \times g_k^{-1}(k_n)(V_1 \times \dots \times V_k) \neq \emptyset.$$

That is,  $g_i^{-1}(k_n)V_i \cap U_i \neq \emptyset$  or  $V_i \cap g_i(k_n)U_i \neq \emptyset$  for all  $i = 1, \dots, k$ .

Then for  $i = 1, \dots, k$ ,

$$g_i(k_n)U_i \cap V_i = g_i(k_n)T^{-n}V_i^{(n-1)} \cap V_i \neq \emptyset.$$

Let

$$V_i^{(n)} = V_i^{(n-1)} \cap (g_i(k_n)T^{-n})^{-1}V_i.$$

Then  $V_i^{(n)} \subseteq V_i^{(n-1)}$  is a non-empty open set and clearly

$$g_i(k_n)T^{-n}V_i^{(n)} \subseteq V_i.$$

Since  $V_i^{(n)} \subseteq V_i^{(n-1)}$ , (2.1) still holds for  $V_i^{(n)}$ . Hence we finish our induction. The proof of the lemma is completed.  $\square$



*Remark 2.7.* By the proof of Lemma 2.6, if for all non-empty open sets  $U_1, \dots, U_d$  and  $V_1, \dots, V_d$  of  $X$ , the set  $\{n \in \mathbb{Z} : U_1 \times \dots \times U_k \cap g_1^{-1}(n) \times \dots \times g_k^{-1}(n)(V_1 \times \dots \times V_d) \neq \emptyset\}$  contains infinitely many positive integers, then we may require that  $\{k_n\}_{n=0}^\infty \subseteq \mathbb{N}$  in Lemma 2.6.

#### 2.3.4. $\{g_1, \dots, g_k\}_\Delta$ -syndetic transitivity and $\{g_1, \dots, g_k\}$ -thickly-syndetic transitivity.

We will need the following definitions.

**Definition 2.8.** We say  $(X, \Gamma)$  is

- (1)  $\{g_1, \dots, g_k\}_\Delta$ -syndetic transitive, if for any given non-empty open sets  $U, V_1, \dots, V_k$  of  $X$ ,

$$\{n \in \mathbb{Z} : U \cap (g_1(n)^{-1}V_1 \cap \dots \cap g_k(n)^{-1}V_k) \neq \emptyset\}$$

is a syndetic set.

- (2)  $\{g_1, \dots, g_k\}$ -thickly-syndetic transitive if for all non-empty open sets  $U_1, \dots, U_k$  and  $V_1, \dots, V_k$  of  $X$ ,

$$\{n \in \mathbb{Z} : U_1 \times \dots \times U_k \cap g_1(n)^{-1} \times \dots \times g_k(n)^{-1}(V_1 \times \dots \times V_k) \neq \emptyset\}$$

is a thickly-syndetic set.

It is clear that  $\{g_1, \dots, g_k\}_\Delta$ -syndetic transitivity implies  $\{g_1, \dots, g_k\}_\Delta$ -transitivity, and  $\{g_1, \dots, g_k\}$ -thickly-syndetic transitivity implies  $\{g_1, \dots, g_k\}$ -transitivity.

### 3. NILPOTENT GROUPS AND PET-INDUCTION

To prove Theorem 1.3, we need some basic results on nilpotent groups and PET-induction. In this section, we cite the basic results related to nilpotent groups from [18], which will be needed in the inductive part of the proof of Theorem 1.3.

In the sequel, let  $\Gamma$  denote a finitely generated nilpotent group without torsion.

#### 3.1. Malcev basis.

**Theorem 3.1.** [18] *Let  $\Gamma$  be a finitely generated nilpotent group without torsion. Then there exists a set of elements  $\{S_1, \dots, S_s\}$  of  $\Gamma$  (the so called, "Malcev basis") such that:*

- (1) *for any  $1 \leq i < j \leq s$ ,  $[S_i, S_j]$  belongs to the subgroup of  $\Gamma$  generated by  $S_1, \dots, S_{i-1}$ ;*
- (2) *every element  $T$  of  $\Gamma$  can be uniquely represented in the form*

$$T = S_1^{r_1(T)} \dots S_s^{r_s(T)}, \quad r_j(T) \in \mathbb{Z}, \quad j = 1, \dots, s;$$

*the mapping  $r : \Gamma \rightarrow \mathbb{Z}^s$ ,  $r(T) = (r_1(T), \dots, r_s(T))$ , being polynomial in the following sense: there exist polynomial mappings  $R : \mathbb{Z}^{2s} \rightarrow \mathbb{Z}^s$ ,  $R' : \mathbb{Z}^{s+1} \rightarrow \mathbb{Z}^s$  such that, for any  $T, T' \in \Gamma$  and any  $n \in \mathbb{N}$ ,*

$$r(TT') = R(r(T), r(T')), \quad r(T^n) = R'(r(T), n).$$

From now on we will fix a Malcev basis  $\{S_1, \dots, S_s\}$  of  $\Gamma$ .



### 3.2. The $\Gamma$ -polynomial group.

An *integral polynomial* is a polynomial taking integer values at the integers.

The group  $\mathbf{P}\Gamma$  is the minimal subgroup of the group  $\Gamma^{\mathbb{Z}}$  of the mappings  $\mathbb{Z} \rightarrow \Gamma$  which contains the constant mappings and is closed with respect to raising to integral polynomials powers: if  $g, h \in \mathbf{P}\Gamma$  and  $p$  is an integral polynomial, then  $gh \in \mathbf{P}\Gamma$ , where  $gh(n) = g(n)h(n)$ , and  $g^p \in \mathbf{P}\Gamma$ , where  $g^p(n) = g(n)^{p(n)}$ . The elements of  $\mathbf{P}\Gamma$  are called  $\Gamma$ -polynomials.  $\Gamma$  itself is a subgroup of  $\mathbf{P}\Gamma$  and is presented by the constant  $\Gamma$ -polynomials.

$\Gamma$ -polynomials taking the value  $e_\Gamma$  at zero form a subgroup of  $\mathbf{P}\Gamma$ ; we denote it by  $\mathbf{P}\Gamma_0$ :  $\mathbf{P}\Gamma_0 = \{g \in \mathbf{P}\Gamma : g(0) = e_\Gamma\}$ . Let  $\mathbf{P}\Gamma_0^* = \{g \in \mathbf{P}\Gamma_0 : g \neq e_\Gamma\}$ .

Every  $\Gamma$ -polynomial  $g$  can be uniquely represented in the form

$$(3.1) \quad g(n) = \prod_{j=1}^s S_j^{p_j(n)} = S_1^{p_1(n)} S_2^{p_2(n)} \dots S_s^{p_s(n)},$$

where  $p_1, \dots, p_s$  are integral polynomials. If  $g \in \mathbf{P}\Gamma_0$  then  $p_1, \dots, p_s \in \mathcal{P}_0$  by Theorem 3.1(2).

### 3.3. The weight of $\Gamma$ -polynomials.

The *weight*,  $w(g)$ , of a  $\Gamma$ -polynomial  $g(n) = \prod_{j=1}^s S_j^{p_j(n)}$  is the pair  $(l, k)$ ,  $l \in \{0, 1, \dots, s\}$ ,  $k \in \mathbb{Z}_+$  for which  $p_j = 0$  for any  $j > l$  and, if  $l \neq 0$ , then  $p_l \neq 0$  and  $\deg(p_l) = k$ . A weight  $(l, k)$  is greater than a weight  $(l', k')$ , denoted by  $(l, k) > (l', k')$ , if  $l > l'$  or  $l = l'$ ,  $k > k'$ .

For example,  $S_1^n, S_1^{n^2} S_2^{n^3}, S_1^{n^6} S_2^{n^6}$  have weights  $(1, 1), (2, 3), (2, 6)$  respectively, and  $(2, 6) > (2, 3) > (1, 1)$ .

Let us now define an equivalence relation on  $\mathbf{P}\Gamma$ :  $g(n) = \prod_{j=1}^s S_j^{p_j(n)}$  is equivalent to  $h(n) = \prod_{j=1}^s S_j^{q_j(n)}$ , if  $w(g) = w(h)$  and, if it is  $(l, k)$ , the leading coefficients of the polynomials  $p_l$  and  $q_l$  coincide; we write then  $g \sim h$ . For example,

$$S_1^n S_3^{n^2} \sim S_3^{n^2+9n} \sim S_1^{n^{12}} S_2^{3n} S_3^{n^2+n}.$$

The *weight* of an equivalence class is the weight of any of its elements.

### 3.4. System and its weight vector.

A *system*  $A$  is a finite subset of  $\mathbf{P}\Gamma$ . For a system  $A$ , if we write  $A = \{f_i\}_{i=1}^v$  then we require that  $f_i \neq f_j$  for  $1 \leq i \neq j \leq v$ . For every system  $A$  we define its *weight vector*  $\phi(A)$  as follows. Let  $w_1 < w_2 < \dots < w_q$  be the set of the distinct weights of all equivalence classes appeared in  $A$ . For  $i = 1, 2, \dots, q$ , let  $\phi(w_i)$  be the number of the equivalence classes of elements of  $A$  with the weight  $w_i$ . Let the weight vector  $\phi(A)$  be

$$\phi(A) = (\phi(w_1)w_1, \phi(w_2)w_2, \dots, \phi(w_q)w_q).$$

For example, let  $A = \{S_1^n, S_1^{2n}, S_1^{n^2}, S_1^n S_2^{2n^2}, S_1^{n^3+n^2} S_2^{2n^2+n}, S_1^{n^5} S_2^{2n^2+2n}, S_1^{n^3} S_2^{2n^2+8n}, S_1^{n^9+n^5+n} S_2^{n^6+n^2}, S_2^{2n^6+n^2}, S_1^n S_2^{3n^6+n^2}\}$ . Then  $\phi(A) = (2(1, 1), 1(1, 2), 1(2, 2), 3(2, 6))$ .

Let  $A, A'$  be two systems. We say that  $A'$  *precedes* a system  $A$  if there exists a weight  $w$  such that  $\phi(A)(w) > \phi(A')(w)$  and  $\phi(A)(u) = \phi(A')(u)$  for all weight  $u > w$ . We denote it by  $\phi(A) \succ \phi(A')$  or  $\phi(A') \prec \phi(A)$ .

For example, let  $w_1 < w_2 < \dots < w_q$  be a sequence of weights, then

$$(a_1 w_1, \dots, a_q w_q) \succ (b_1 w_1, \dots, b_q w_q)$$

if and only if  $(a_1, \dots, a_q) > (b_1, \dots, b_q)$ .

### 3.5. PET-induction.

In order to prove that a result holds for all systems  $A$ , we start with the system whose weight vector is  $\{1(1, 1)\}$ . That is,  $A = \{S_1^{c_1 n}\}$ , where  $c_1 \in \mathbb{Z} \setminus \{0\}$ . Then let  $A \subseteq \mathbf{P}\Gamma$  be a system whose weight vector is greater than  $\{1(1, 1)\}$ , and assume that for all systems  $A'$  preceding  $A$ , we have that the result holds for  $A'$ . Once we show that the result still holds for  $A$ , we complete the whole proof. This procedure is called the *PET-induction*.

## 4. OUTLINE OF THE PROOF OF THEOREM 1.2

To show the general ideas of the proof of Theorem 1.3, in this section we outline the general idea how to prove Theorem 1.2. Since we deal with only one transformation in Theorem 1.2, it is relatively easy to present the basic ideas of the proof and see how PET-induction works. In Section 5, we will give the complete proof of Theorem 1.3.

4.1. Throughout this section,  $p_1, \dots, p_d \in \mathcal{P}_0^*$  are distinct polynomials, and  $(X, T)$  is a weakly mixing minimal system. By Lemma 2.4, it suffices to show that  $(X, T)$  is  $\{(T^{p_1(n)}, \dots, T^{p_d(n)})\}_\Delta$ -transitive. And in fact we will prove a stronger result:

- If  $p_1, \dots, p_d \in \mathcal{P}_0^*$  are distinct polynomials, then  $(X, T)$  is  $\{T^{p_1(n)}, \dots, T^{p_d(n)}\}$ -thickly-syndetic transitive, and it is  $\{T^{p_1(n)}, \dots, T^{p_d(n)}\}_\Delta$ -syndetic transitive.

### 4.2. The PET-induction.

4.2.1. Now  $\Gamma = \mathbb{Z} = \langle T \rangle$ , and  $\mathbf{P}\Gamma = \{T^{p(n)} : p \in \mathcal{P}\}$ . For each  $T^{p(n)} \in \mathbf{P}\Gamma$ , its weight  $w(T^{p(n)}) = (1, k)$ , where  $k$  is the degree of  $p(n)$ . A system  $A$  has the form of  $\{T^{p_1(n)}, T^{p_2(n)}, \dots, T^{p_d(n)}\}$ , where  $p_1, \dots, p_d \in \mathcal{P}$  are distinct polynomials. Its weight vector  $\phi(A)$  has the form of

$$(a_1(1, 1), a_2(1, 2), \dots, a_k(1, k)).$$

For example, the weight vector of  $\{T^{c_1 n}, \dots, T^{c_m n}\}$  is  $(m(1, 1))$  if  $c_1, \dots, c_m$  are distinct and non-zero; the weight vector of  $\{T^{an^2+b_1 n}, \dots, T^{an^2+b_d n}\}$  ( $a \neq 0$ ) is  $(1(1, 2))$ ; and the weight vector of  $\{T^{an^2+b_1 n}, \dots, T^{an^2+b_d n}, T^{c_1 n}, \dots, T^{c_m n}\}$  ( $a \neq 0$  and  $c_1, \dots, c_m$  are distinct and non-zero) is  $(m(1, 1), 1(1, 2))$ ; and the weight vector of the general polynomials of degree  $\leq 2$  is  $(m(1, 1), k(1, 2))$ .

Under the order of weight vectors, one has

$$\begin{aligned} (1(1, 1)) &< (2(1, 1)) < \dots < (m(1, 1)) < \dots < (1(1, 2)) < (1(1, 1), 1(1, 2)) < \dots < \\ &(m(1, 1), 1(1, 2)) < \dots < (2(1, 2)) < (1(1, 1), 2(1, 2)) < \dots < (m(1, 1), 2(1, 2)) < \dots < \\ &(m(1, 1), k(1, 2)) < \dots < (1(1, 3)) < (1(1, 1), 1(1, 3)) < \dots < (m(1, 1), k(1, 2), 1(1, 3)) \\ &< \dots < (2(1, 3)) < \dots < (a_1(1, 1), a_2(1, 2), \dots, a_k(1, k)) < \dots \end{aligned}$$

4.2.2. To prove Theorem 1.2, we will use induction on the weight vectors. We start from the systems with the weight vector  $(1(1, 1))$ , i.e.  $A = \{T^{a_1 n}\}$ . After that, we assume that the result holds for all systems whose weight vectors are  $< (a_1(1, 1), a_2(1, 2), \dots, a_k(1, k))$ . Then we show that the result also holds for the system with weight vector  $(a_1(1, 1), a_2(1, 2), \dots, a_k(1, k))$ , and hence the proof is completed.

To illustrate the basic ideas, we show the result for the system  $A = \{T^{n^2}, T^{2n^2}\}$ , whose weight vector is  $(2(1, 2))$ . The general proof of Theorem 1.2 is similar, and we omit it here. We will give the details in the proof of Theorem 1.3.

4.3. **Example:**  $(X, T)$  is  $\{T^{n^2}, T^{2n^2}\}_\Delta$ -transitive.

To show this example, we need to verify the following cases one by one:

**Case 1** when the weight vector is  $(d(1, 1))$ :  $(X, T)$  is  $\{T^{a_1 n}, \dots, T^{a_d n}\}_\Delta$ -syndetic transitive, where  $a_1, \dots, a_d \in \mathbb{Z} \setminus \{0\}$  are distinct integers.

**Case 2** when the weight vector is  $(1(1, 2))$ :

- (1)  $(X, T)$  is  $\{T^{an^2+b_1 n}, \dots, T^{an^2+b_d n}\}$ -thickly-syndetic transitive,
- (2)  $(X, T)$  is  $\{T^{an^2+b_1 n}, \dots, T^{an^2+b_d n}\}_\Delta$ -syndetic transitive, where  $b_1, \dots, b_d$  are distinct integers and  $a \in \mathbb{Z} \setminus \{0\}$ .

**Case 3** when the weight vector is  $(r(1, 1), 1(1, 2))$ :

- (1)  $(X, T)$  is  $\{T^{c_1 n}, \dots, T^{c_r n}, T^{an^2+b_1 n}, \dots, T^{an^2+b_d n}\}$ -thickly-syndetic transitive,
- (2)  $(X, T)$  is  $\{T^{c_1 n}, \dots, T^{c_r n}, T^{an^2+b_1 n}, \dots, T^{an^2+b_d n}\}_\Delta$ -syndetic transitive, where  $a \in \mathbb{Z} \setminus \{0\}$ ,  $b_1, \dots, b_d$  are distinct integers and  $c_1, \dots, c_r$  are distinct non-zero integers.

**Case 4** when the weight vector is  $(2(1, 2))$ :

- (1)  $(X, T)$  is  $\{T^{n^2}, T^{2n^2}\}$ -thickly-syndetic transitive,
- (2)  $(X, T)$  is  $\{T^{n^2}, T^{2n^2}\}_\Delta$ -transitive.

4.3.1. *Case 1:*  $(X, T)$  is  $\{T^{a_1 n}, \dots, T^{a_d n}\}_\Delta$ -syndetic transitive, where  $a_1, \dots, a_d$  are distinct non-zero integers.

*Proof.* We will prove Case 1 by induction on  $d$ . By Lemma 2.1, Case 1 holds for  $d = 1$ . Now we assume that the result holds for  $d \geq 1$ . That is, for any non-empty open subsets  $U, V_1, \dots, V_d$  of  $X$  and for distinct non-zero integers  $c_1, \dots, c_d$ ,

$$\{n \in \mathbb{Z} : U \cap T^{-c_1 n} V_1 \cap \dots \cap T^{-c_d n} V_d \neq \emptyset\}$$

is a syndetic set.

Now let  $U, V_1, \dots, V_d, V_{d+1}$  be non-empty open subsets of  $X$  and  $a_1, \dots, a_d, a_{d+1}$  are distinct non-zero integers. We will show that

$$N := \{n \in \mathbb{Z} : U \cap T^{-a_1 n} V_1 \cap \dots \cap T^{-a_d n} V_d \cap T^{-a_{d+1} n} V_{d+1} \neq \emptyset\}$$

is syndetic. Write  $p_1(n) = a_1 n, \dots, p_{d+1}(n) = a_{d+1} n$ .

Since  $(X, T)$  is minimal, there is some  $\ell \in \mathbb{N}$  such that  $X = \bigcup_{j=0}^{\ell} T^j U$ .

By Corollary 2.2,  $(X^{d+1}, T^{a_1} \times \dots \times T^{a_{d+1}})$  is weakly mixing. By Lemma 2.6, there are non-empty subsets  $V_1^{(\ell)}, \dots, V_{d+1}^{(\ell)}$  and integers  $k_0, k_1, \dots, k_\ell$  such that for each  $i = 1, 2, \dots, d+1$ , one has that

$$T^{p_i(k_j)} T^{-j} V_i^{(\ell)} \subseteq V_i, \quad \text{for all } 0 \leq j \leq \ell.$$

Let  $q_1(n) = p_2(n) - p_1(n) = (a_2 - a_1)n, \dots, q_d(n) = p_{d+1}(n) - p_1(n) = (a_{d+1} - a_1)n$ . Since  $a_2 - a_1, \dots, a_{d+1} - a_1$  are distinct non-zero integers, by the induction hypothesis,

$$E = \{n \in \mathbb{Z} : V_1^{(\ell)} \cap T^{-q_1(n)} V_2^{(\ell)} \cap \dots \cap T^{-q_d(n)} V_{d+1}^{(\ell)} \neq \emptyset\}$$

is syndetic.

Let  $m \in E$ . Then there is some  $x_m \in V_1^{(\ell)}$  such that  $T^{q_i(m)} x_m \in V_{i+1}^{(\ell)}$  for  $i = 1, \dots, d$ . Clearly, there is some  $y_m \in X$  with  $y_m = T^{-p_1(m)} x$ . Since  $X = \bigcup_{j=0}^{\ell} T^j U$ , there is some  $b_m \in \{0, 1, \dots, \ell\}$  such that  $T^{b_m} z_m = y_m$  for some  $z_m \in U$ . Thus for each  $i = 1, 2, \dots, d+1$

$$\begin{aligned} T^{p_i(m+k_{b_m})} z_m &= T^{p_i(m+k_{b_m})} T^{-b_m} y_m = T^{p_i(m+k_{b_m})} T^{-b_m} T^{-p_1(m)} x_m \\ &= T^{p_i(k_{b_m})} T^{-b_m} T^{p_i(m)-p_1(m)} x_m \\ &= T^{p_i(k_{b_m})} T^{-b_m} T^{q_{i-1}(m)} x_m \quad (\text{Let } q_0(n) = 0) \\ &\in T^{p_i(k_{b_m})} T^{-b_m} V_i^{(\ell)} \subseteq V_i. \end{aligned}$$

That is,

$$z_m \in U \cap T^{-p_1(n)} V_1 \cap \dots \cap T^{-p_d(n)} V_d \cap T^{-p_{d+1}(n)} V_{d+1},$$

where  $n = m + k_{b_m}$ . Thus

$$N \supseteq \{m + k_{b_m} : m \in E\}$$

is a syndetic set. By induction the proof is completed.  $\square$

*Remark 4.1.* Note that Case 1 is the strengthened version of Glasner's theorem.

4.3.2. *Case 2:* (1)  $(X, T)$  is  $\{T^{an^2+b_1n}, \dots, T^{an^2+b_dn}\}$ -thickly-syndetic transitive, where  $b_1, \dots, b_d$  are distinct integers and  $a \in \mathbb{Z} \setminus \{0\}$ .

*Proof.* Since the family of thickly-syndetic sets is a filter, it suffices to show that for any  $p(n) = an^2 + bn$  ( $a \neq 0, a, b \in \mathbb{Z}$ ), one has that for all non-empty open sets  $U, V \subseteq X$

$$N_p(U, V) = \{n \in \mathbb{Z} : U \cap T^{-p(n)} V \neq \emptyset\}$$

is thickly-syndetic.

Since  $(X, T)$  is minimal, there is some  $\ell \in \mathbb{N}$  such that  $X = \bigcup_{i=0}^{\ell} T^i U$ .

Let  $L \in \mathbb{N}$  and let  $k_i = i(L+2)$  for all  $i \in \{0, 1, \dots, \ell\}$ . Since  $(X, T)$  is weakly mixing and minimal, by Lemma 2.1

$$C := \bigcap_{(i,j) \in \{0,1,\dots,\ell\} \times \{0,1,\dots,L\}} \{k \in \mathbb{Z} : V \cap T^{-k} (T^{p(k_i+j)-i})^{-1} V \neq \emptyset\}$$

is a thickly-syndetic set. Choose  $c \in C$ . Then for any  $(i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$  one has

$$V_{i,j} := V \cap (T^{p(k_i+j)+c-i})^{-1} V$$

is a non-empty open subset of  $V$  and

$$T^{p(k_i+j)+c-i}V_{i,j} \subset V.$$

Let  $p_{i,j}(n) = p(k_i+j+n) - p(k_i+j) - p(n) = 2ak_in + 2ajn$  for any  $(i,j) \in \{0,1,\dots,\ell\} \times \{0,1,\dots,L\}$ . Since  $k_i = i(L+2)$ ,  $p_{i,j}$  are distinct for all  $(i,j) \in \{0,1,\dots,\ell\} \times \{0,1,\dots,L\}$ . By Case 1,

$$D := \{n \in \mathbb{Z} : V \cap \bigcap_{(i,j) \in \{0,1,\dots,\ell\} \times \{0,1,\dots,L\}} T^{-p_{i,j}(n)}V_{i,j} \neq \emptyset\}$$

is a syndetic set.

For  $m \in D$ , there exists  $x_m \in V$  such that  $T^{p_{i,j}(m)}x_m \in V_{i,j}$  for any  $(i,j) \in \{0,1,\dots,\ell\} \times \{0,1,\dots,L\}$ . Let  $y_m = T^{-p(m)}x_m$ . Since  $X = \bigcup_{i=0}^{\ell} T^i U$ , there are  $z_m \in U$  and  $0 \leq b_m \leq \ell$  such that  $T^c y_m = T^{b_m} z_m$ . Then  $z_m = T^{-p(m)+c-b_m}x_m$  and we have

$$\begin{aligned} T^{p(m+k_{b_m}+j)}z_m &= T^{p(m+k_{b_m}+j)}T^{-p(m)+c-b_m}x_m \\ &= T^{p(k_{b_m}+j)+c-b_m}(T^{p(k_{b_m}+j+m)-p(k_{b_m}+j)-p(m)}x_m) \\ &= T^{p(k_{b_m}+j)+c-b_m}(T^{p_{b_m,j}(m)}x_m) \\ &\in T^{p(k_{b_m}+j)+c-b_m}V_{b_m,j} \subset V \end{aligned}$$

for each for  $j \in \{0,1,\dots,L\}$ . Thus

$$\{m+k_{b_m}+j : 0 \leq j \leq L\} \subset N_p(U, V).$$

Hence the set  $\{n \in \mathbb{Z} : n+j \in N_p(U, V) \text{ for any } j \in \{0,1,\dots,L\}\}$  contains the syndetic set  $\{m+k_{b_m} : m \in D\}$ . As  $L \in \mathbb{N}$  is arbitrary,  $N_p(U, V)$  is a thickly-syndetic set.  $\square$

4.3.3. Case 2: (2)  $(X, T)$  is  $\{T^{an^2+b_1n}, \dots, T^{an^2+b_dn}\}_{\Delta}$ -syndetic transitive, where  $b_1, \dots, b_d$  are distinct integers and  $a \in \mathbb{Z} \setminus \{0\}$ .

*Proof.* Let  $p_1(n) = an^2 + b_1n, \dots, p_d(n) = an^2 + b_dn$ . We will show for any given non-empty open subsets  $U, V_1, \dots, V_d$  of  $X$

$$N = \{n \in \mathbb{Z} : U \cap (T^{-p_1(n)}V_1 \cap \dots \cap T^{-p_d(n)}V_d) \neq \emptyset\}$$

is syndetic.

Since  $(X, T)$  is minimal, there is some  $\ell \in \mathbb{N}$  such that  $X = \bigcup_{i=0}^{\ell} T^i U$ . Then by Case 2(1) and Lemma 2.6 there are integers  $\{k_j\}_{j=0}^{\ell}$  and non-empty open sets  $V_i^{(\ell)} \subset V_i$ ,  $1 \leq i \leq d$  such that  $|k_j| > |k_{j-1}| + \sum_{i=1}^d |b_i|$  for  $j = 0, \dots, \ell$  ( $k_{-1} = 0$ ) and

$$T^{p_i(k_j)}T^{-j}V_i^{(\ell)} \subset V_i, \quad 0 \leq j \leq \ell, 1 \leq i \leq d.$$

Let  $q_i(m, n) = p_i(n+m) - p_i(m) - p_1(n)$  for  $n, m \in \mathbb{Z}$  and  $i = 1, \dots, d$ . Since  $|k_j| > |k_{j-1}| + \sum_{i=1}^d |b_i|$  for  $j = 0, \dots, \ell$ , we have that all  $q_i(k_j, n)$  are distinct polynomials in  $n$  with degree 1 for  $0 \leq j \leq \ell, 1 \leq i \leq d$ . By Case 1,

$$E = \{n \in \mathbb{Z} : V_1^{(\ell)} \cap \bigcap_{j=1}^{\ell} \left( T^{-q_1(k_j, n)}V_1^{(\ell)} \cap \dots \cap T^{-q_d(k_j, n)}V_d^{(\ell)} \right) \neq \emptyset\}$$

is syndetic.

Let  $m \in E$ . Then there is some  $x_m \in V_1^{(\ell)}$  such that

$$T^{q_i(k_j, m)} x_m \in V_i^{(\ell)} \text{ for all } 1 \leq i \leq d \text{ and } 0 \leq j \leq \ell.$$

Clearly, there is some  $y_m \in X$  such that  $y_m = T^{-p_1(m)} x$ . Since  $X = \bigcup_{j=0}^{\ell} T^j U$ , there is some  $b_m \in \{0, 1, \dots, \ell\}$  such that  $T^{b_m} z_m = y_m$  for some  $z_m \in U$ . Thus for each  $i = 1, 2, \dots, d$

$$\begin{aligned} T^{p_i(m+k_{b_m})} z_m &= T^{p_i(m+k_{b_m})} T^{-b_m} y_m = T^{p_i(m+k_{b_m})} T^{-b_m} T^{-p_1(m)} x_m \\ &= T^{p_i(k_{b_m})} T^{-b_m} T^{p_i(m+k_{b_m})-p_i(k_{b_m})-p_1(m)} x_m \\ &= T^{p_i(k_{b_m})} T^{-b_m} T^{q_i(k_{b_m}, m)} x_m \\ &\in T^{p_i(k_{b_m})} T^{-b_m} V_i^{(\ell)} \subseteq V_i. \end{aligned}$$

That is,

$$z_m \in U \cap T^{-p_1(n)} V_1 \cap \dots \cap T^{-p_d(n)} V_d,$$

where  $n = m + k_{b_m}$ . Thus

$$N \supseteq \{m + k_{b_m} : m \in E\}$$

is a syndetic set. The proof is completed.  $\square$

**4.3.4. Case 3: (1)**  $(X, T)$  is  $\{T^{c_1 n}, \dots, T^{c_r n}, T^{an^2+b_1 n}, \dots, T^{an^2+b_d n}\}$ -thickly-syndetic transitive, where  $a \in \mathbb{Z} \setminus \{0\}$ ,  $b_1, \dots, b_d$  are distinct integers and  $c_1, \dots, c_r$  are distinct non-zero integers.

*Proof.* It follows from Case 2(1) and Lemma 2.1.  $\square$

**4.3.5. Case 3: (2)**  $(X, T)$  is  $\{T^{c_1 n}, \dots, T^{c_r n}, T^{an^2+b_1 n}, \dots, T^{an^2+b_d n}\}_{\Delta}$ -syndetic transitive, where  $a \in \mathbb{Z} \setminus \{0\}$ ,  $b_1, \dots, b_d$  are distinct integers and  $c_1, \dots, c_r$  are distinct non-zero integers.

*Proof.* The proof is almost the same to the proof of Case 2 (2). The only difference is that we need to deal with it by induction on  $r$ .

Let  $p_1(n) = c_1 n, \dots, p_r(n) = c_r n, p_{r+1}(n) = an^2 + b_1 n, \dots, p_{r+d}(n) = an^2 + b_d n$ . We will show for any given non-empty open sets  $U, V_1, \dots, V_t$  (where  $t = r + d$ )

$$N = \{n \in \mathbb{Z} : U \cap (T^{-p_1(n)} V_1 \cap \dots \cap T^{-p_t(n)} V_t) \neq \emptyset\}$$

is syndetic.

Since  $(X, T)$  is minimal, there is some  $\ell \in \mathbb{N}$  such that  $X = \bigcup_{i=0}^{\ell} T^i U$ . Then by Case 3(1) and Lemma 2.6 there are integers  $\{k_j\}_{j=0}^{\ell}$  and non-empty open sets  $V_i^{(\ell)} \subset V_i$ ,  $1 \leq i \leq t$  such that  $|k_j| > |k_{j-1}| + \sum_{i=1}^d |b_i|$  for  $j = 0, \dots, \ell$  (here  $k_{-1} = 0$ ) and

$$T^{p_i(k_j)} T^{-j} V_i^{(\ell)} \subset V_i, \quad 0 \leq j \leq \ell, 1 \leq i \leq t.$$

Let  $q_i(m, n) = p_i(n + m) - p_i(m) - p_1(n)$  for  $n, m \in \mathbb{Z}$  and  $i = 1, \dots, t$ . Then

$$q_i(k_j, n) = (c_i - c_1)n$$

for  $n \in \mathbb{Z}$ ,  $1 \leq i \leq r$  and  $0 \leq j \leq \ell$ . Since  $|k_j| > |k_{j-1}| + \sum_{i=1}^d |b_i|$  for  $j = 0, \dots, \ell$ , we have that all  $q_{r+i}(k_j, n) = an^2 + (2ak_j + b_i - c_1)n$  are distinct polynomials in  $n$  with degree 2 for  $0 \leq j \leq \ell, 1 \leq i \leq d$ .

Note that  $q_1(k_j, n) = 0$  for  $n \in \mathbb{Z}$  and  $0 \leq j \leq \ell$ . By Case 2(2) if  $r = 1$ , or by the inductive assumption if  $r \geq 2$ ,

$$\begin{aligned} E &= \{n \in \mathbb{Z} : V_1^{(\ell)} \cap \bigcap_{j=1}^{\ell} \left( T^{-q_1(k_j, n)} V_1^{(\ell)} \cap \dots \cap T^{-q_t(k_j, n)} V_t^{(\ell)} \right) \neq \emptyset\} \\ &= \{n \in \mathbb{Z} : V_1^{(\ell)} \cap \left( \bigcap_{i=2}^r T^{-(c_i - c_1)n} V_i^{(\ell)} \right) \cap \bigcap_{j=1}^{\ell} \left( \bigcap_{i=1}^d T^{-q_{r+i}(k_j, n)} V_{r+i}^{(\ell)} \right) \neq \emptyset\} \end{aligned}$$

is syndetic. The rest of proof is the same to the proof in Case 2(2).  $\square$

4.3.6. Case 4: (1)  $(X, T)$  is  $\{T^{n^2}, T^{2n^2}\}$ -thickly-syndetic transitive.

*Proof.* It follows from Case 2(1).  $\square$

4.3.7. Case 4: (2)  $(X, T)$  is  $\{T^{n^2}, T^{2n^2}\}_{\Delta}$ -transitive.

*Proof.* The proof is almost the same to the proof of Case 2(2). Let  $p_1(n) = n^2$ ,  $p_2(n) = 2n^2$ . We will show for any given non-empty open subsets  $U, V_1, V_2$  of  $X$

$$N = \{n \in \mathbb{Z} : U \cap (T^{-p_1(n)} V_1 \cap T^{-p_2(n)} V_2) \neq \emptyset\}$$

is syndetic.

Since  $(X, T)$  is minimal, there is some  $\ell \in \mathbb{N}$  such that  $X = \bigcup_{i=0}^{\ell} T^i U$ . Then by Case 4(1) and Lemma 2.6 there are integers  $\{k_j\}_{j=0}^{\ell}$  and non-empty open sets  $V_i^{(\ell)} \subset V_i$ ,  $1 \leq i \leq \ell$  such that  $|k_j| > |k_{j-1}|$  for  $j = 0, \dots, \ell$  (here  $k_{-1} = 0$ ) and

$$T^{p_i(k_j)} T^{-j} V_i^{(\ell)} \subset V_i, \quad 1 \leq j \leq \ell, 1 \leq i \leq 2.$$

Let  $q_i(m, n) = p_i(n + m) - p_i(m) - p_1(n)$  for  $n, m \in \mathbb{Z}$  and  $i = 1, 2$ . Since  $\{|k_j|\}$  is an increasing sequence of natural numbers, we have that all

$$q_i(k_j, n) = \begin{cases} 2k_j n & \text{if } i = 1 \\ n^2 + 2k_j n & \text{if } i = 2 \end{cases}$$

are distinct non-constant polynomials in  $n$  for  $0 \leq j \leq \ell, 1 \leq i \leq 2$ . By Case 3(2),

$$E = \{n \in \mathbb{Z} : V_1^{(\ell)} \cap \bigcap_{j=1}^{\ell} \left( T^{-q_1(k_j, n)} V_1^{(\ell)} \cap T^{-q_2(k_j, n)} V_2^{(\ell)} \right) \neq \emptyset\}$$

is syndetic. The same proof to the Case 2(2), for all  $m \in E$  one finds some  $b_m \in \{0, \dots, \ell\}$  such that  $m + k_{b_m} \in N$ , and hence

$$N \supseteq \{m + k_{b_m} : m \in E\}$$

is a syndetic set. The proof is completed.  $\square$

## 5. PROOF OF THEOREMS 1.3 AND 1.4

In this section, we give a proof of Theorems 1.3 and 1.4.



5.1. Let  $(X, \Gamma)$  be a t.d.s., where  $\Gamma$  is a nilpotent group such that for each  $T \in \Gamma$ ,  $T \neq e_\Gamma$ , is weakly mixing and minimal. Thus,  $\Gamma$  is a nilpotent group without torsion. For  $d, k \in \mathbb{N}$  let  $T_1, \dots, T_d \in \Gamma$ , and  $p_{i,j} \in \mathcal{P}_0$ ,  $1 \leq i \leq k$ ,  $1 \leq j \leq d$  such that the expressions

$$g_i(n) = T_1^{p_{i,1}(n)} \dots T_d^{p_{i,d}(n)}$$

depends nontrivially on  $n$  for  $i = 1, 2, \dots, k$ , and for all  $i \neq j \in \{1, 2, \dots, k\}$  the expressions  $g_i(n)g_j(n)^{-1}$  depend nontrivially on  $n$ .

By Lemma 2.4, to prove Theorem 1.3 it remains to show that for any given non-empty open sets  $U, V_1, \dots, V_k$  of  $X$  there is  $n \in \mathbb{N}$  such that

$$U \cap (g_1(n)^{-1}V_1 \cap \dots \cap g_k(n)^{-1}V_k) \neq \emptyset,$$

i.e.  $(X, \Gamma)$  is  $A_\Delta$ -transitive. Moreover, we also need to show it is  $A$ -thickly-syndetic transitive in the same time, where  $A = \{g_1, \dots, g_k\}$ .

## 5.2. Some lemmas.

### 5.2.1. Some basic results by Leibman.

**Lemma 5.1.** [18, Lemma 2.4.] *Let  $g$  be a  $\Gamma$ -polynomial.*

- (1) *If  $h$  is a  $\Gamma$ -polynomial and  $g' = h^{-1}gh$ , then  $g' \sim g$ .*
- (2) *If  $m \in \mathbb{N}$  and  $g'$  is defined by  $g'(n) = g^{-1}(m)g(n+m)$ , then  $g' \sim g$ .*
- (3) (a) *If  $g', h$  are  $\Gamma$ -polynomials such that  $g' \sim g$ ,  $h \not\sim g$  and  $w(h) \preceq w(g)$ , then  $g'h^{-1} \sim gh^{-1}$  and  $w(gh^{-1}) = w(g)$*   
 (b) *If  $h \neq e_\Gamma$  is a  $\Gamma$ -polynomial such that  $h \sim g$ , then  $w(gh^{-1}) \prec w(g)$ .*

**Corollary 5.2.** [18, Corollary 2.5.] *Let  $A$  be a system.*

- (1) *If  $A'$  is a system consisting of  $\Gamma$ -polynomials of the form  $g' = h^{-1}gh$  for  $g \in A$  and  $h$  being a  $\Gamma$ -polynomial, then  $\phi(A') \preceq \phi(A)$ .*
- (2) *If  $A'$  is a system consisting of  $\Gamma$ -polynomials  $g'$  satisfying the equality  $g'(n) = g^{-1}(m)g(n+m)$  for some  $g \in A$  and some  $m \in \mathbb{N}$ , then  $\phi(A') \preceq \phi(A)$ .*
- (3) *Let  $h \in A, h \neq e_\Gamma$ , be a  $\Gamma$ -polynomial of weight minimal in  $A$ :  $w(h) \leq w(g)$  for any  $g \in A$ . If  $A'$  is a system consisting of  $\Gamma$ -polynomials of the form  $g' = gh^{-1}, g \in A$ , then  $\phi(A') \prec \phi(A)$ .*

5.2.2. *Additional lemmas.* To show the main result we find that above lemma and corollary are not enough. We need some additional lemmas which we shall prove in this subsection.

Using (3.1) and Theorem 3.1(2), it is clear that for  $\Gamma$ -polynomial  $g$ , if

$$\{n \in \mathbb{Z} : g(n) = e_\Gamma\}$$

is an infinite set then  $g \equiv e_\Gamma$  since every non-zero integral polynomial has finitely many zero points. In fact if  $|\{n \in \mathbb{Z} : g(n) = e_\Gamma\}| > k$  for some  $k$  depending only on  $g$ , then  $g \equiv e_\Gamma$ .

**Lemma 5.3.** *Let  $f, g \in \mathbf{P}\Gamma_0$ . Then*

- (1) *If  $\{k' \in \mathbb{Z} : f(k')^{-1}f(n+k') = g(n) \text{ for all } n \in \mathbb{Z}\}$  is an infinite set, then  $g(n) = f(n) = (f(1))^n$  for all  $n \in \mathbb{Z}$ .*

(2) If  $\{k' \in \mathbb{Z} : f(k')^{-1}f(n+k') = g(k')^{-1}g(n+k') \text{ for all } n \in \mathbb{Z}\}$  is an infinite set, then  $g = f$ .

*Proof.* 1). Let  $E = \{k' \in \mathbb{Z} : f(k')^{-1}f(n+k') = g(n) \text{ for all } n \in \mathbb{Z}\}$ . For  $n \in \mathbb{Z}$ , we consider the  $\Gamma$ -polynomial  $p_n(k) = f(k)^{-1}f(n+k)g(n)^{-1}$  with respect to  $k$ . Since  $E \subseteq \{k \in \mathbb{Z} : p_n(k) = e_\Gamma\}$ , one has  $p_n(k) = e_\Gamma$  for all  $k \in \mathbb{Z}$ . This implies  $f(k)^{-1}f(n+k) = g(n)$  for all  $n, k \in \mathbb{N}$ . Note that  $f(0) = e_\Gamma$ , so  $f(n) = g(n)$  for all  $n \in \mathbb{Z}$ . Then it follows from the equation  $f(n+k) = f(n)f(k)$  for all  $n, k \in \mathbb{Z}$ , one has that

$$f(n) = g(n) = f(1)^n, \text{ for all } n \in \mathbb{Z}.$$

2). Let  $F = \{k' \in \mathbb{Z} : f(k')^{-1}f(n+k') = g(k')^{-1}g(n+k') \text{ for all } n \in \mathbb{Z}\}$ . For  $n \in \mathbb{Z}$ , we consider the polynomial  $q_n(k) = f(k)^{-1}f(n+k)(g(k)^{-1}g(n+k))^{-1}$  with respect to  $k$ . Since  $F \subseteq \{k \in \mathbb{Z} : q_n(k) = e_\Gamma\}$ , one has  $q_n(k) = e_\Gamma$  for all  $k \in \mathbb{Z}$ . This implies  $f(k)^{-1}f(n+k) = g(k)^{-1}g(n+k)$  for all  $n, k \in \mathbb{N}$ . Since  $f(0) = e_\Gamma$  and  $g(0) = e_\Gamma$ , one has that  $f(n) = g(n)$  for all  $n \in \mathbb{Z}$ .  $\square$

**Lemma 5.4.** Let  $f \in \mathbf{PF}_0$ . If for each  $m \in \mathbb{Z} \setminus \{0\}$  there is some  $n = n(m) \in \mathbb{Z}$  such that  $f(m+n) \neq f(m)f(n)$ , then for any  $\ell, L \in \mathbb{N}$  we can find  $k_0, k_1, \dots, k_\ell \in \mathbb{N}$  such that

- (1)  $f(k_i + j)^{-1}f(k_i + j + n)f(n)^{-1} \in \mathbf{PF}_0^*$  for any  $(i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$ .
- (2)  $f(k_i + j)^{-1}f(k_i + j + n)f(n)^{-1}$  and  $f(k_{i'} + j')^{-1}f(k_{i'} + j' + n)f(n)^{-1}$  are distinct  $\Gamma$ -polynomials with respect to  $n$  for any  $(i, j), (i', j') \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$  with  $(i, j) \neq (i', j')$ .

*Proof.* Let

$$K = \{k \in \mathbb{Z} : f(k)^{-1}f(n+k) = f(n) \text{ for all } n \in \mathbb{Z}\}.$$

Then by Lemma 5.3(1),  $K$  is a finite set. Fix  $\ell, L \in \mathbb{N}$ . For  $(i, j), (i', j') \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$  with  $(i, j) \neq (i', j')$ , we denote  $E((i, j), (i', j'))$  by the set of all  $k \in \mathbb{Z}$  satisfying that

$$f(i(L+2)+k+j)^{-1}f(i(L+2)+k+j+n) = f(i'(L+2)+k+j')^{-1}f(i'(L+2)+k+j'+n)$$

for all  $n \in \mathbb{Z}$ .

We claim that  $E((i, j), (i', j'))$  is a finite set. Assume the contrary that  $E((i, j), (i', j'))$  is not a finite set. Let  $m = (i' - i)(L+2) + (j' - j)$ . Since  $(i, j) \neq (i', j')$ , one has  $m \neq 0$ . Put  $g(n) = f(m)^{-1}f(n+m)$  for  $n \in \mathbb{Z}$ . Then  $g \in \mathbf{PF}_0$  and  $g \neq f$  by the assumption of the lemma. Note that

$$g(i(L+2)+k+j)^{-1}g(i(L+2)+k+j+n) = f(i'(L+2)+k+j')^{-1}f(i'(L+2)+k+j'+n)$$

for all  $n \in \mathbb{Z}$ . We have

$$\begin{aligned} & \{k \in \mathbb{Z} : f(k)^{-1}f(k+n) = g(k)^{-1}g(n+k) \text{ for all } n \in \mathbb{Z}\} \\ &= E((i, j), (i', j')) + (i(L+2) + j) \end{aligned}$$

is an infinite set. Thus by Lemma 5.3(2) one has that  $f = g$ , a contradiction! This shows that  $E((i, j), (i', j'))$  is a finite set.

Set

$$E = \bigcup_{\substack{(i,j) \neq (i',j') \\ \in \{0,1,\dots,\ell\} \times \{0,1,\dots,L\}}} E((i, j), (i', j')).$$

Then  $E$  is also a finite set. Put

$$F = E \cup \{k - (i(L+2) + j) : k \in K, (i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}\}.$$

It is clear that  $F$  is finite.

Now we take  $u \in \mathbb{N} \setminus F$ . Let  $k_i = i(L+2) + u$  for  $i \in \{0, 1, \dots, \ell\}$ . On one hand, for any  $(i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$  one has  $f(k_i + j)^{-1}f(k_i + j + n)f(n)^{-1} \in \mathbf{P}\Gamma_0^*$  as  $k_i + j \notin K$ . On the other hand, for  $(i, j), (i', j') \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$  with  $(i, j) \neq (i', j')$  one has that  $f(k_i + j)^{-1}f(k_i + j + n)f(n)^{-1}$  and  $f(k_{i'} + j')^{-1}f(k_{i'} + j' + n)f(n)^{-1}$  are distinct  $\Gamma$ -polynomials with respect to  $n$  since  $u \notin E((i, j), (i', j'))$ . Thus we finish the proof of the lemma.  $\square$

**Lemma 5.5.** *Let  $f_1, f_2, \dots, f_v \in \mathbf{P}\Gamma_0^*$  be distinct  $\Gamma$ -polynomials. Then there exists a sequence  $\{r(i)\}_{i=0}^\infty$  of natural numbers such that for any  $\ell \in \mathbb{N}$  and  $k_0, k_1, \dots, k_\ell \in \mathbb{N}$  with  $k_0 > r(0)$  and  $k_i > k_{i-1} + r(k_{i-1})$  for  $i = 1, \dots, \ell$ , one has that*

- (1)  $f_t(k_i)^{-1}f_t(n + k_i)f_1(n)^{-1} \in \mathbf{P}\Gamma_0^*$  for any  $t \in \{2, \dots, v\}$  and  $i \in \{0, 1, \dots, \ell\}$ .
- (2)  $f_t(k_i)^{-1}f_t(n + k_i)f_1(n)^{-1}$  and  $f_s(k_j)^{-1}f_s(n + k_j)f_1(n)^{-1}$  are distinct  $\Gamma$ -polynomial with respect to  $n$  for any  $t \neq s \in \{1, 2, \dots, v\}$  and  $i, j \in \{0, 1, \dots, \ell\}$ .

*Proof.* First  $f_t(k)^{-1}f_t(n + k)f_1(n)^{-1} \in \mathbf{P}\Gamma_0$  for any  $t \in \{1, 2, \dots, v\}$  and  $k \in \mathbb{Z}$ . Then by Lemma 5.3 (1), for  $t \in \{2, \dots, v\}$

$$K_t := \{k \in \mathbb{Z} : f_t(k)^{-1}f_t(n + k) = f_1(n) \text{ for all } n \in \mathbb{Z}\}$$

is a finite set since  $f_1 \neq f_t$ . Thus for  $t \in \{2, \dots, v\}$  we may take  $L_t \in \mathbb{N}$  such that  $K_t \subseteq [-L_t, L_t]$ .

For any  $t, s \in \{1, 2, \dots, v\}$  and  $k' \in \mathbb{Z}$ , we put

$$K_{t,s}(k') := \{k \in \mathbb{Z} : f_t(k)^{-1}f_t(n + k) = f_s(k')^{-1}f_s(n + k') \text{ for all } n \in \mathbb{Z}\}.$$

If  $K_{t,s}(k')$  is an infinite set then by Lemma 5.3 (1) one has

$$f_s(k')^{-1}f_s(n + k') = f_t(n) = (f_t(1))^n$$

for all  $n \in \mathbb{Z}$ . Take  $n = -k'$  one has  $f_s(k') = (f_t(1))^{k'}$  as  $f_s(0) = e_\Gamma$ . Thus

$$f_s(m) = f_s(k')f_t(m - k') = (f_t(1))^{k'}(f_t(1))^{m - k'} = (f_t(1))^m = f_t(m)$$

for all  $m \in \mathbb{Z}$ . Hence  $f_s = f_t$ . This implies  $s = t$ .

The above discussion shows that  $K_{t,s}(k')$  is a finite set for any  $t \neq s \in \{1, 2, \dots, v\}$  and  $k' \in \mathbb{Z}$ . Thus for any  $t \neq s \in \{1, 2, \dots, v\}$  and  $k' \in \mathbb{Z}$  we may take  $L_{t,s}(k') \in \mathbb{N}$  such that  $K_{t,s}(k') \subseteq [-L_{t,s}(k'), L_{t,s}(k')]$ .

Next by Lemma 5.3 (2), for  $t \neq s \in \{1, 2, \dots, v\}$

$$K_{t,s} := \{k \in \mathbb{Z} : f_t(k)^{-1}f_t(n + k) = f_s(k)^{-1}f_s(n + k) \text{ for all } n \in \mathbb{Z}\}$$

is a finite set since  $f_t \neq f_s$ . Thus for  $t \neq s \in \{1, 2, \dots, v\}$  we may take  $L_{t,s} \in \mathbb{N}$  such that  $K_{t,s} \subseteq [-L_{t,s}, L_{t,s}]$ .

For  $i \geq 0$ , we take

$$r(i) = 1 + \max_{t \in \{2, \dots, v\}} L_t + \max_{\substack{t \neq s \in \{1, \dots, v\}, \\ k' \in \{0, 1, \dots, i\}}} (L_{t,s} + L_{t,s}(k')).$$

Now for any given  $\ell \in \mathbb{N}$  and  $k_0, k_1, \dots, k_\ell \in \mathbb{N}$  with  $k_0 > r(0)$  and  $k_i > k_{i-1} + r(k_{i-1})$  for  $i = 1, \dots, \ell$ , on the one hand for any  $t \in \{2, \dots, v\}$  and  $i \in \{0, 1, \dots, \ell\}$  one has  $f_t(k_i)^{-1} f_t(n+k_i) f_1(n)^{-1} \in \mathbf{PI}_0^*$  since  $k_i \notin K_t$ . On the other hand for any  $t \neq s \in \{1, 2, \dots, v\}$  and  $0 \leq i \leq j \leq \ell$  one has  $f_t(k_i)^{-1} f_t(n+k_i)$  and  $f_s(k_j)^{-1} f_s(n+k_j)$  are distinct  $\Gamma$ -polynomials with respect to  $n$  as  $k_j \notin K_{t,s}$  and  $k_j \notin \bigcup_{0 \leq r \leq j-1} K_{t,s}(k_r)$ . This clearly implies that  $f_t(k_i)^{-1} f_t(n+k_i) f_1(n)^{-1}$  and  $f_s(k_j)^{-1} f_s(n+k_j) f_1(n)^{-1}$  are distinct. We finish the proof of the lemma.  $\square$

### 5.3. Proof of Theorem 1.3.

We will prove Theorem 1.3 using the PET-induction introduced in Section 3. We will use the notations in Section 3 freely. Recall that  $A = \{g_1, \dots, g_k\}$ .

We start with the system whose weight vector is  $\{1(1, 1)\}$ . That is,  $A = \{S_1^{c_1}\}$ , where  $c_1 \in \mathbb{Z} \setminus \{0\}$ . Since  $S_1^{c_1} \neq e_\Gamma$ ,  $(X, S_1^{c_1})$  is weakly mixing and minimal. By Lemma 2.1 for any non-empty open sets  $U_1$  and  $V_1$  of  $X$ ,

$$\{n \in \mathbb{Z} : U_1 \cap S_1^{-c_1 n} V_1 \neq \emptyset\}$$

is a thickly-syndetic set. Hence  $X$  is  $A$ -thickly-syndetic transitive and  $A_\Delta$ -syndetic transitive.

Now let  $A \subseteq \mathbf{PI}_0^*$  be a system whose weight vector is greater than  $\{1(1, 1)\}$ , and assume that for all systems  $A'$  preceding  $A$ , we have  $(X, \Gamma)$  is  $A'$ -thickly-syndetic transitive and  $A'_\Delta$ -syndetic transitive. Now we show that  $(X, \Gamma)$  is  $A$ -thickly-syndetic transitive and  $A_\Delta$ -syndetic transitive.

5.3.1. *Claim:  $(X, \Gamma)$  is  $A$ -thickly-syndetic transitive.*

Since the intersection set of two thickly-syndetic subsets is still a thickly-syndetic subset, it is sufficient to show that for any  $f \in A$ , and for any given non-empty open subsets  $U, V$  of  $X$ ,

$$N_f(U, V) := \{n \in \mathbb{Z} : U \cap f(n)^{-1} V \neq \emptyset\}$$

is a thickly-syndetic set.

Let  $T \in \Gamma$  be an element in the center of  $\Gamma$  with  $T \neq e_\Gamma$ . As  $(X, T)$  is minimal there is  $\ell \in \mathbb{N}$  such that  $X = \bigcup_{i=0}^{\ell} T^i U$ . For given  $f \in A$  and non-empty open subsets  $U, V$  of  $X$ , we have the following two cases.

**Case 1:** The first case is that there exists  $m \in \mathbb{Z} \setminus \{0\}$  such that  $f(n+m) = f(m)f(n)$  for all  $n \in \mathbb{Z}$ . Thus  $f(-m) = (f(m))^{-1}$  and  $f(n-m) = f(-m)f(n)$  for all  $n \in \mathbb{Z}$ . Let  $u = |m|$ . Then  $u > 0$  and

$$f(n+u) = f(u)f(n) \text{ for all } n \in \mathbb{Z}.$$

This implies that for  $r = 0, 1, \dots, u-1$

$$f(ku+r) = (f(u))^k f(r) \text{ for all } k \in \mathbb{Z}.$$

Since  $f \neq e_\Gamma$  and  $f(0) = e_\Gamma$ , one has that  $f(u) \neq e_\Gamma$ . Thus  $(X, f(u))$  is weakly mixing and minimal. Hence

$$B_r := \{k \in \mathbb{Z} : f(r)U \cap (f(u))^{-k} V \neq \emptyset\}$$

is a thickly-syndetic subset of  $\mathbb{Z}$ .

Put  $B = \bigcap_{0 \leq r \leq u-1} B_r$ . Then  $B$  is a thickly-syndetic subset of  $\mathbb{Z}$ . Note that

$$N_f(U, V) \supseteq \bigcup_{0 \leq r \leq u-1} \{ku + r : k \in B_r\} \supseteq \{ku + r : k \in B, r \in \{0, 1, \dots, u-1\}\}.$$

Thus  $N_f(U, V)$  is a thickly-syndetic subset of  $\mathbb{Z}$  as  $B$  is a thickly-syndetic subset of  $\mathbb{Z}$ .

**Case 2:** The second case is that for each  $m \in \mathbb{Z} \setminus \{0\}$  there is some  $n = n(m) \in \mathbb{Z}$  such that  $f(m+n) \neq f(m)f(n)$ . Fix  $L \in \mathbb{N}$ . By Lemma 5.4 for any  $\ell, L \in \mathbb{N}$  we can find  $k_0, k_1, \dots, k_\ell \in \mathbb{N}$  such that

- (1)  $f(k_i + j)^{-1}f(k_i + j + n)f(n)^{-1} \in \mathbf{PI}_0^*$  for any  $(i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$ .
- (2)  $f(k_i + j)^{-1}f(k_i + j + n)f(n)^{-1}$  and  $f(k_{i'} + j')^{-1}f(k_{i'} + j' + n)f(n)^{-1}$  are distinct  $\Gamma$ -polynomials with respect to  $n$  for any  $(i, j), (i', j') \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$  with  $(i, j) \neq (i', j')$ .

Since  $(X, T)$  is weakly mixing and minimal,

$$C := \bigcap_{(i,j) \in \{0,1,\dots,\ell\} \times \{0,1,\dots,L\}} \{k \in \mathbb{Z} : V \cap T^{-k}((f(k_i + j)T^{-i})^{-1}V) \neq \emptyset\}$$

is a thickly-syndetic set. Choose  $a \in C$ . Then for any  $(i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$  one has

$$V_{i,j} := V \cap (f(k_i + j)T^{a-i})^{-1}V$$

is a non-empty open subset of  $V$  and

$$f(k_i + j)T^{a-i}V_{i,j} \subset V.$$

Write  $p_{i,j}(n) = f(k_i + j)^{-1}f(k_i + j + n)f(n)^{-1}$  for any  $(i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$ . Let

$$A^L = \{p_{i,j} : (i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}\}.$$

Then  $A^L \subset \mathbf{PI}_0^*$  is a system and by Corollary 5.2

$$\phi(A^L) \prec \phi(\{f\}) \preceq \phi(A).$$

Hence  $A^L$  precedes  $A$ . By the inductive assumption,  $X$  is  $A_\Delta^L$ -syndetic transitive. Thus

$$D := \{n \in \mathbb{Z} : V \cap \bigcap_{(i,j) \in \{0,1,\dots,\ell\} \times \{0,1,\dots,L\}} (p_{i,j}(n))^{-1}V_{i,j} \neq \emptyset\}$$

is a syndetic set.

For  $m \in D$ , there exists  $x_m \in V$  such that  $p_{i,j}(m)x_m \in V_{i,j}$  for any  $(i, j) \in \{0, 1, \dots, \ell\} \times \{0, 1, \dots, L\}$ . Let  $y_m = f(m)^{-1}x_m$ . Since  $X = \bigcup_{i=0}^\ell T^i U$ , there are  $z_m \in U$  and  $0 \leq b_m \leq \ell$  such that  $T^a y_m = T^{b_m} z_m$ . Then  $z_m = f(m)^{-1}T^{a-b_m}x_m$  and we have

$$\begin{aligned} f(m + k_{b_m} + j)z_m &= f(m + k_{b_m} + j)f(m)^{-1}T^{a-b_m}x_m \\ &= f(k_{b_m} + j)T^{a-b_m}(f(k_{b_m} + j)^{-1}f(k_{b_m} + j + m)f(m)^{-1}x_m) \\ &= f(k_{b_m} + j)T^{a-b_m}(p_{k_{b_m},j}(m)x_m) \\ &\in f(k_{b_m} + j)T^{a-b_m}V_{b_m,j} \subset V \end{aligned}$$

for each for  $j \in \{0, 1, \dots, L\}$ . Thus

$$\{m + k_{b_m} + j : 0 \leq j \leq L\} \subset N_f(U, V).$$

Hence the set  $\{n \in \mathbb{Z} : n + j \in N_f(U, V) \text{ for any } j \in \{0, 1, \dots, L\}\}$  contains the syndetic set  $\{m + k_{b_m} : m \in D\}$ . As  $L \in \mathbb{N}$  is arbitrary,  $N_f(U, V)$  is a thickly-syndetic subset of  $\mathbb{Z}$ .

5.3.2. *Claim:  $(X, \Gamma)$  is  $A_\Delta$ -syndetic transitive.*

Let  $A = \{f_1, \dots, f_v\}$ . Then  $f_1, f_2, \dots, f_v$  are distinct  $\Gamma$ -polynomials. It remains to prove that for any given non-empty open sets  $U, V_1, \dots, V_v$  of  $X$

$$(5.1) \quad N_A(U, V_1, \dots, V_v) := \{n \in \mathbb{Z} : U \cap f_1(n)^{-1}V_1 \cap \dots \cap f_v(n)^{-1}V_v \neq \emptyset\}$$

is a syndetic set.

Let  $T \in \Gamma$  be an element from the center of  $\Gamma$  with  $T \neq e_\Gamma$ . As  $(X, T)$  is minimal, there is some  $\ell \in \mathbb{N}$  such that  $X = \bigcup_{i=0}^{\ell} T^i U$ . Let  $f \in A, f \neq e_\Gamma$ , be a  $\Gamma$ -polynomial of weight minimal in  $A$ :  $w(f) \leq w(f_j)$  for any  $j = 1, \dots, v$ . Without loss of generality assume that  $f = f_1$ . By Lemma 2.6 and Lemma 5.5, there are  $\{k_j\}_{j=0}^{\ell} \subseteq \mathbb{N}$  and  $V_t^{(\ell)} \subset V_t$  for  $t = 1, \dots, v$  such that

- (1) For  $t = 1, \dots, v, f_t(k_j)T^{-j}V_t^{(\ell)} \subset V_t, \quad \forall 0 \leq j \leq \ell$ .
- (2)  $f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1}$  and  $f_s(k_i)^{-1}f_s(n+k_i)f(n)^{-1}$  are distinct  $\Gamma$ -polynomials with respect to  $n$  for any  $t \neq s \in \{1, 2, \dots, v\}$  and  $i, j \in \{0, 1, \dots, \ell\}$ .
- (3)  $f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1} \in \mathbf{P}\Gamma_0^*$  for any  $t \in \{2, \dots, v\}$  and  $j \in \{0, 1, \dots, \ell\}$ .

Since it may happen that for some  $t$  there are  $i \neq j$  with

$$f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1} = f_t(k_i)^{-1}f_t(n+k_i)f(n)^{-1}$$

for all  $n \in \mathbb{Z}$ , for each  $t \in \{1, 2, \dots, v\}$  let  $I_t \subset \{0, 1, \dots, \ell\}$  such that the elements of  $\{f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1} : j \in I_t\}$  are distinct  $\Gamma$ -polynomial in  $\mathbf{P}\Gamma_0^*$  and

$$\{f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1} : j \in I_t\} = \{f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1} : j = 0, 1, \dots, \ell\} \setminus \{e_\Gamma(n)\},$$

where  $e_\Gamma(n)$  is the constant  $\Gamma$ -polynomial with value  $e_\Gamma$ . Note that  $I_1 = \emptyset$  if and only if  $f_1(k_j)^{-1}f_1(n+k_j)f(n)^{-1} \equiv e_\Gamma$  for each  $j \in \{0, 1, \dots, \ell\}$ . Moreover, by the above condition (3),  $|I_t| \geq 1$  and

$$\{f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1} : j \in I_t\} = \{f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1} : j = 0, 1, \dots, \ell\}$$

for any  $t \geq 2$ .

Let

$$A' = \bigcup_{t=1}^v \{f_t(k_j)^{-1}f_t(n+k_j)f(n)^{-1} : j \in I_t\}.$$

Then  $A' \subseteq \mathbf{P}\Gamma_0^*$  and by Corollary 5.2,  $A'$  precedes  $A$ . According to the inductive assumption,  $X$  is  $A'_\Delta$ -syndetic transitive. Hence

$$E := \{m \in \mathbb{Z} : V_1^{(\ell)} \cap \bigcap_{t=1}^v \bigcap_{j \in I_t} (f_t(k_j)^{-1}f_t(m+k_j)f(m)^{-1})^{-1}V_t^{(\ell)} \neq \emptyset\}$$

is a syndetic set.

For  $m \in E$ , there are  $x_m \in V_1^{(\ell)}$  such that

$$f_t(k_i)^{-1} f_t(m + k_i) f(m)^{-1} x_m \in V_t^{(\ell)}, \text{ for any } 1 \leq t \leq v, i \in I_t.$$

Moreover by the choice of  $I_t$  and  $x_m \in V_1^{(\ell)}$ , one has that

$$f_t(k_j)^{-1} f_t(m + k_j) f(m)^{-1} x_m \in V_t^{(\ell)}, \text{ for any } 1 \leq t \leq v, j \in \{0, 1, \dots, \ell\}.$$

Let  $y_m = f(m)^{-1} x_m$ . Since  $X = \bigcup_{i=0}^{\ell} T^i U$ , there is  $z_m \in U$  and  $0 \leq b_m \leq \ell$  such that  $y_m = T^{b_m} z_m$ . Then  $z_m = T^{-b_m} f(m)^{-1} x_m$  and we have

$$\begin{aligned} f_t(m + k_{b_m}) z_m &= f_t(m + k_{b_m}) T^{-b_m} f(m)^{-1} x_m \\ &= f_t(k_{b_m}) T^{-b_m} (f_t(k_{b_m})^{-1} f_t(m + k_{b_m}) f(m)^{-1}) x_m \\ &\in f_t(k_{b_m}) T^{-b_m} V_t^{(\ell)} \subset V_t \end{aligned}$$

for each  $1 \leq t \leq v$ . This implies that

$$z_m \in U \cap f_1(n)^{-1} V_1 \cap \dots \cap f_v(n)^{-1} V_v$$

with  $n = m + k_{b_m}$ . Thus

$$N_A(U, V_1, \dots, V_v) \supseteq \{m + k_{b_m} : m \in E\}$$

is a syndetic set. Hence the proof of the whole theorem is completed.

#### 5.4. Proof of Theorem 1.4.

We have the following two cases:

**Case 1:**  $g(n) = (g(1))^n$  for any  $n \in \mathbb{Z}$ . Then  $g(1) \neq e_\Gamma$  as  $g \not\equiv e_\Gamma$ . Since  $(X, g(1))$  is minimal, for each  $x \in X$  and each non-empty open subset  $U$  of  $X$ ,  $\{n \in \mathbb{Z} : g(n)x \in U\}$  is syndetic.

**Case 2:** There exists  $v \in \mathbb{Z}$  such that  $g(v) \neq (g(1))^v$ . Thus  $g(u+1) \neq g(u)g(1)$  for some  $u \in \mathbb{Z}$ . Let  $f(n) = g(n)^{-1} g(n+1) g(1)^{-1}$  for  $n \in \mathbb{Z}$ . Then  $f(u) \neq e_\Gamma$  and so  $f \in \mathbf{P}\Gamma_0^*$ .

Assume that the weight of the  $\Gamma$ -polynomial  $f(n) = \prod_{j=1}^s S_j^{p_j(n)}$  is  $(l, k)$ . Then  $k > 0$  and the degree of the integral polynomial  $p_\ell$  is  $k$  as  $f \in \mathbf{P}\Gamma_0^*$ . Thus there exists  $M \in \mathbb{N}$  such that  $p_\ell$  is strictly monotone on  $[M, +\infty)$ . Particularly, for any  $i, j \geq M$  with  $i \neq j$  we have  $p_\ell(i) \neq p_\ell(j)$  and hence  $f(i) \neq f(j)$  by Theorem 3.1(2).

By Theorem 1.3 for each given non-empty open subset  $V$  of  $X$ ,

$$F =: \{n \in \mathbb{Z} : V \cap g(n)^{-1} V \neq \emptyset\} = \{\dots < n_{-1} < n_0 < n_1 < \dots\}$$

is thickly syndetic, where we require  $n_0 \geq M$ . Since  $F$  is syndetic, there is  $L(V) \in \mathbb{N}$  such that

$$n_{i+1} - n_i \leq L(V)$$

for  $i \in \mathbb{N}$ . Consider  $\Gamma$ -polynomials  $\{g(n_i)^{-1} g(n + n_i) : i \in \mathbb{Z}\}$ . Since  $g \in \mathbf{P}\Gamma_0^*$ ,

$$g^{-1}(n_i) g(n + n_i) \in \mathbf{P}\Gamma_0^*$$

for any  $i \in \mathbb{Z}$ .



Now for any  $i, j \in \mathbb{N}$  with  $i \neq j$ , note that

$$g(n_i)^{-1}g(1+n_i) = f(n_i)g(1) \neq f(n_j)g(1) = g(n_j)^{-1}g(1+n_j),$$

hence  $g(n_i)^{-1}g(1+n_i)$  and  $g(n_j)^{-1}g(1+n_j)$  are distinct  $\Gamma$ -polynomials in  $\mathbf{P}\Gamma_0^*$ .

For  $d \in \mathbb{N}$ , let  $A_d(V)$  be the set of all points  $y_d \in X$  such that we can find  $m_1 < \dots < m_d \in \mathbb{Z}$  satisfying  $g(m_j)y_d \in V$  for  $j = 1, \dots, d$  and  $m_{j+1} < m_j + L(V)$  for  $j = 1, \dots, d-1$ .

For any  $i \in \mathbb{Z}$ , let  $V_i = V \cap g(n_i)^{-1}V$ . Then  $V_i$  is a non-empty open subset of  $V$  and  $g(n_i)V_i \subset V$ . Let  $U$  be a non-empty open subset of  $X$ . Applying Theorem 1.1, there are  $y_d \in U$  and  $l_d \in \mathbb{Z}$  such that  $g^{-1}(n_j)g(l_d+n_j)y_d \in V_j$  which implies that  $g(l_d+n_j)y_d \in V$  for  $j = 1, \dots, d$ . Thus  $y_d \in U \cap A_d(V)$ .

By what we just proved,  $A_d(V)$  is an open dense subset of  $X$  as  $U$  is arbitrary. Assume that  $\{U_i\}$  is a base of the topology of  $X$  and  $X_0 = \bigcap_{i \in \mathbb{N}} \bigcap_{d \in \mathbb{N}} A_d(U_i)$ . We claim  $X_0$  is the set we need. In fact, for any non-empty open subset  $U$  of  $X$ , there is  $i \in \mathbb{N}$  with  $U_i \subset U$ . So for  $x \in X_0$ ,  $x \in \bigcap_{d \in \mathbb{N}} A_d(U_i)$ . Thus, for any  $d \in \mathbb{N}$ , there are  $m_1^d < m_2^d < \dots < m_d^d \in \mathbb{Z}$  such that  $m_{j+1}^d - m_j^d \leq L(U_i)$  for  $j = 1, \dots, d-1$  and

$$N_g(x, U) \supset N_g(x, U_i) \supset \bigcup_{d \in \mathbb{N}} \{m_1^d, m_2^d, \dots, m_d^d\}$$

i.e.  $\{n \in \mathbb{Z} : g(n)x \in U\}$  is piecewise syndetic.

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